Partially coherent solitons of variable shape in a slow Kerr-like medium: Exact solutions

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We carry out a theoretical investigation of the properties of partially coherent solitons for media which have a slow Kerr-like nonlinearity. We find exact solutions of the Nth-order Manakov equations in a general form. These describe partially coherent solitons (PCSs) and their collisions. In fact, the exact solutions allow us to analyze important properties of PCSs such as stationary profiles of the spatial beams and effects resulting from their collisions. In particular, we find, analytically, the number of parameters that control the soliton shape. We present profiles which are symmetric as well as those which are asymmetric. We also find that collisions allow the profiles to remain stationary but cause their shapes to change. [S1063-651X(99)08705-X]

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I. INTRODUCTION

The theory of self-action of incoherent light beams is a relatively old subject [1,2]. In the temporal domain, the notion of temporal incoherent solitons was introduced by Hasegawa in a series of works [3–5], both for plasma waves and for nonlinear pulses in multimode fibers. However, the creation of incoherent solitons in optical fibers requires unrealistically high pulse energies. Photorefractive materials are probably the only media for experimental studies of incoherent beams, as they generally exhibit very strong nonlinear effects with extremely low optical powers [6–9]. The problem of spatial incoherent solitons began to attract a great deal of attention only recently [10–16], after an experimental observation of partially incoherent solitons (PCSs) was made by Mitchell *et al.* [17,18]. The experiment was done with photorefractive material with a drift nonlinearity where coherent photorefractive solitons had been found to exist earlier [19,20].

There are a few different approaches to a theoretical investigation of incoherent solitons. The most direct approach is based on the equation for the field correlation function [1,2]. A description of spatial incoherent solitons, based on the ''coherent density approach,'' where the partially coherent beam is represented as a superposition of mutually incoherent components, has been developed by Christodoulides *et al.* [10,11]. For the special case of the logarithmic nonlinearity, the symmetric solutions can be written in analytic form [10].

The description of a partially coherent stationary soliton as a multimode self-induced waveguide [12–15] has been especially fruitful. The main idea is that the modes must be self-consistent with the soliton profile, as in the case of higher-order solitons [21,22]. Then stationary soliton propagation can be obtained by adjusting the amplitudes of various mutually incoherent linear modes of the self-induced waveguide. Due to mutual incoherence, the total light intensity is a direct sum of the intensities of all excited modes. Thus, mode beating, which is a feature of coherent interaction, is absent. On the other hand, while the qualitative approach is useful in producing numerical results, it lacks the generality required for finding all possible exact solutions, and consequently completing analysis of the problem. At this point we should note that there is a slight difference in the methods used to create PCSs in the cases considered in [12,15] and [13,14]. In the first instance, the PCSs are formed by superimposing a few cw mutually incoherent optical beams. In the second case, various components of the partially coherent solitons are derived from an incoherent light source of finite extent. However, as has been pointed out in [16], these two configurations are completely equivalent as far as propagation in a slowly responding medium is concerned.

A diffractionless ray optics limit for treating spatial incoherent solitons has been proposed by Snyder and Mitchell [23]. This approach is accurate when the size of the PCS is much larger than the optical wavelength. In terms of a multimode waveguide, this limit is valid when the number of modes goes to infinity, so that the soliton becomes completely incoherent. This approach is useful for wide beams. However, all intermediate cases must be covered as well.

Most of the above-referenced works only showed the existence of symmetric solutions for PCSs. On the other hand, in the works [3] [the case of one dimensional (1D) solitons in Kerr media] and [23] (the case of 3D solitons in media with arbitrary nonlinearity) it was pointed out that incoherent solitons may have arbitrary shapes in the regime of complete incoherence. This controversy has been resolved in [24]. It has been shown that a PCS can be considered simultaneously as a self-induced waveguide and also as a multisoliton complex. This complementarity in viewing PCSs greatly enhances our understanding of PCSs and their properties. In the present work, we further develop the theory of PCSs. Namely, we show *analytically* that PCSs can have profiles which are variable and which are governed by a finite number of parameters. The number of parameters depends on the number of linear modes comprising the PCS. At one extreme, when the PCS forms a single-moded waveguide, the soliton is coherent, its shape is symmetric, and it is described by the sech function. This is the case of a single fundamental soliton. At the other extreme, when the number of modes goes to infinity, the number of parameters which control the shape is also infinite. In this limit, the soliton effectively has an arbitrary profile.

The interaction of incoherent and partially coherent solitons is another interesting area of research which has only been addressed in recent papers [14,24]. In fact, the paper [24] gave a complete qualitative description of PCSs and their collisions. Nevertheless, more research is needed to understand the interaction of PCSs in nonlinear media and to describe them analytically.

In this paper, we give a complete description of PCSs using exact solutions of the generalized Manakov equations. This allows us to investigate not only the properties of stationary PCSs, but their collisions as well. The method for finding exact solutions of Schrödinger equations with special potentials was developed a long time ago by Kay and Moses [25]. The starting point was to find potentials which have the property of being reflectionless. It is important that these are solitonlike profiles at the same time. The multiplicity of such potentials has been related to the fact that these are self-consistent potentials for a set of linear equations. Apart from the self-consistency which is required for nonlinear objects, partially coherent solitons are also objects which can be described by the technique [25].

It turns out that a refinement of the method developed by Kay and Moses is suitable for finding solutions of a nonlinear set of ordinary differential equations with cubic nonlinearities [26]. The self-consistency requirement relates linear and cubic nonlinear equations. This gives us the chance to linearize the nonlinear equations and to find solutions which are multisoliton complexes. We use this method to find exact solutions of multiwave equations describing PCSs. Another generalization was made by Nogami and Warke [27]. They presented a method for constructing multisoliton solutions of N coupled nonlinear Schrödinger equations (NLSEs). However, some important features of linear equations have been missed in [27]. An important point to realize here is that the set of functions considered by Nogami and Warke is translationally invariant, thus giving an additional $(N-1)$ parameters to the solutions which relate to our interest. This approach, with some modifications which add more parameters into the solution, allows us not only to find stationary multicomponent solutions for the set of generalized Manakov equations, but also exact solutions for colliding solitons. Using this approach allows us to find solutions for collisions of self-induced waveguides. In particular, we have found exact solutions for the process of collisions of PCSs. These can be treated using the linear set of functional equations which simultaneoulsy give solutions of generalized Manakov equations. Importantly enough, the technique allows us to write solutions for arbitrary N using just one matrix equation.

II. STATEMENT OF THE PROBLEM

It has been shown that propagation of partially coherent wave packets in nonlinear media with a slow nonlinear response can be represented by a set of equations for the mutually incoherent components of the packet [11,3]. For a beam (or beams) consisting of N components, the corresponding equations in media with Kerr-like nonlinearity have the form

$$i\frac{\partial \psi_i}{\partial z}+\frac{1}{2}\frac{\partial^2 \psi_i}{\partial x^2}+\alpha\,\delta n(I)\,\psi_i=0, \tag{1}$$

where ψ_i denotes the ith component of the beam, α is the coefficient representing the strength of nonlinearity, x is the transverse coordinate, z is the coordinate along the direction of propagation, and

$$\delta n(I)=\sum_{i=1}^{N}|\psi_i|^2 \tag{2}$$

is the change in refractive index profile created by all incoherent components of the light beam. The medium is considered to have a slow response, so that the intensities contribute to the change of the refractive index but the relative phases between the components do not.

We are interested in solutions of Eq. (1) in the form of partially coherent solitons. These are stationary waveguides self-induced by their own modes. The self-consistency condition requires that these solutions be multisoliton complexes. Namely, these are nonlinear superpositions of fundamental solitons propagating in parallel and thus creating the waveguide. This complementary view is important for a physical understanding of PCSs. Fortunately, Eq. (1) is integrable and, in principle, all of its solutions can be found in analytical form. The mathematical treatment of the problem also admits this complementarity: the equations can be written either as linear Schrödinger equations for each mode or as a set of nonlinear equations. The latter allows us to treat the solution for partially coherent solitons as a nonlinear superposition of N solitons related to each of the N components, respectively. Let us consider these special solutions.

III. PARTIALLY COHERENT SOLITONS

Stationary solutions of Eq. (1) are given by

$$\psi_i(x,z)=\frac{1}{\sqrt{\alpha}}u_i(x)\exp\left(i\frac{k_i^2}{2}z\right), \tag{3}$$

with real functions $u_i(x)$ and real eigenvalues k_i. Then the set of Eqs. (1) reduces to the set of ODEs:

$$\frac{\partial^2 u_j}{\partial x^2}+2\left[\sum_{i=1}^{N}u_i^2\right]u_j=k_j^2u_j\,. \tag{4}$$

This set of equations is also completely integrable for an arbitrary set of real nondegenerate k_i. Using the results of [26,27], it can be shown that its solutions can be found from the linear set of algebraic equations:

$$\sum_{i=1}^{N}\frac{\exp[k_ix]\exp[k_jx]}{k_j+k_i}\frac{u_i(x)}{\sqrt{2k_i}}+\frac{u_j(x)}{\sqrt{2k_j}}=-\exp[k_jx], \tag{5}$$

which can be written in a matrix form:

$$D_{j,m}\frac{u_m(x)}{\sqrt{2k_m}}=-e_j\,, \tag{6}$$

where the terms in matrix D are

$$D_{j,m}=\delta_{j,m}+\frac{e_j e_m}{k_j+k_m} \tag{7}$$

and

$$e_j=\exp(k_j x). \tag{8}$$

Henceforth, we will replace the e_j functions with more general ones, namely,

$$e_j=\sqrt{2k_j a_j}\exp(k_j\overline{x_j}), \tag{9}$$

where $\overline{x_j}=x-x_j$ and the parameters x_j are shifts for each fundamental soliton. These are parameters which nontrivially contribute to the shape of the PCS. The new functions also give a solution for each u_j. The new feature of the functions e_j here is the addition, not only of shifts x_j, but also of arbitrary coefficients a_i. We could absorb the x_j into the a_j, but we keep both the coefficients a_j and x_j as independent parameters. The reason is that the coefficients a_j define the specific choice needed to achieve symmetry in the presentation of the solution and the x_j define fundamental soliton locations in the multisoliton complex.

We arrange the eigenvalues required in decreasing order ($k_1>k_2>k_3>\cdots$) and define the positive coefficient

$$c_{ij}=\frac{k_i+k_j}{|k_i-k_j|}.$$

It happens that the choice

$$a_i=\prod_{j\neq i} c_{ij} \tag{10}$$

is the one which allows us to obtain the above-mentioned symmetry, provided that all $x_i=0$. Note that each $a_i>0$. For example, if there are four eigenvalues, then

$$a_2=\prod_{j\neq 2} c_{2j}=c_{21}c_{23}c_{24}=c_{12}c_{23}c_{24}.$$

If, on the other hand, the x_i's remain arbitrary parameters, then the solution is asymmetric, but is represented in the same compact and convenient form.

The solution components themselves can be written in a simple form:

$$u_i(x)=-\sqrt{2k_i}D_{i,j}^{-1}e_j, \tag{11}$$

where the vector e_j is also given by Eq. (8). Although the inversion of the matrix D is a standard technique, it requires some effort to present the solution in a compact and simple form. This is essentially what is done in the three following sections.

As we can see from the above discussions, the solution is actually a multiparameter family. It contains N soliton parameters, k_i, as well as N shifts, x_i. Admitting translational symmetry of the solution as a whole, we can define all shifts relative to one of them, so that the total solution then contains $2N-1$ free parameters. These parameters give a huge diversity of PCS shapes. These solutions have been discussed in the literature only partly [13,14]. This means that some rough classification of these solutions is needed. The simplest case is when the relative distances between the solitons are larger than their widths. Then the solution set consists of N well-separated solitons, each as a separate component. When solitons are located close to each other, the solution is more complicated. All of these solutions are stable on propagation.

Below, we will investigate particular cases.

IV. GENERAL (ARBITRARY EIGENVALUES) SOLUTION FOR $N=2$

For $N=1$, we define $D_1=\cosh(k_1\overline{x_1})$, so the fundamental NLSE soliton is $u_1(x)=k_1/D_1=k_1\,\mathrm{sech}(k_1\overline{x_1})$. Let us consider the case $N=2$. The matrix elements are given by

$$D_{11}=1+a_1\exp(2k_1\overline{x_1}),$$

$$D_{22}=1+a_2\exp(2k_2\overline{x_2}),$$

$$D_{12}=D_{21}=\frac{2\sqrt{a_1a_2}}{k_1+k_2}\sqrt{k_1k_2}\exp(k_1\overline{x_1}+k_2\overline{x_2}).$$

The specific choice needed to achieve symmetry is

$$a_2=a_1=c_{12}=\frac{k_1+k_2}{k_1-k_2}. \tag{12}$$

Choosing these special coefficients [Eq. (12)] and inverting the matrix D gives, after some simple algebra,

$$u_1=\pm\frac{2k_1\sqrt{a_1}}{D_2}\cosh(k_2\overline{x_2}), \tag{13}$$

and

$$u_2=\pm\frac{2k_2\sqrt{a_2}}{D_2}\sinh(k_1\overline{x_1}), \tag{14}$$

where

$$D_2=\cosh(k_1\overline{x_1}+k_2\overline{x_2})+c_{12}\cosh(k_1\overline{x_1}-k_2\overline{x_2}). \tag{15}$$

This form of the solution is convenient for generalizations when $N>2$ and can be viewed as the standard form. Other forms have been used in the presentation of this solution in Refs. [28–31] and in [24].

The solution is asymmetric in general for arbitrary k_1 and k_2 but becomes symmetric for the special choice of $\Delta x_{12}=x_2-x_1=0$. Then u_1 and u_2 are, respectively, the even and odd modes of a symmetric self-induced waveguide. If $k_1/k_2=2$ with k_2 arbitrary, then D_2 reduces to $4\cosh^3(k_2x)$ and $u_1^2+u_2^2$ is simply $3k_2^2\,\mathrm{sech}^2(k_2x)$. Figure 1 shows the two modes as well as the intensity profile for two different separations, Δx_{12}. Note that the intensity profile for the symmetric solution does not necessarily have to have a single maximum. When k_1 and k_2 are close to each other, the solution may show two peaks in the intensity profile. An example of double peak structure of a symmetric PCS with $N=2$ is shown in Fig. 6.

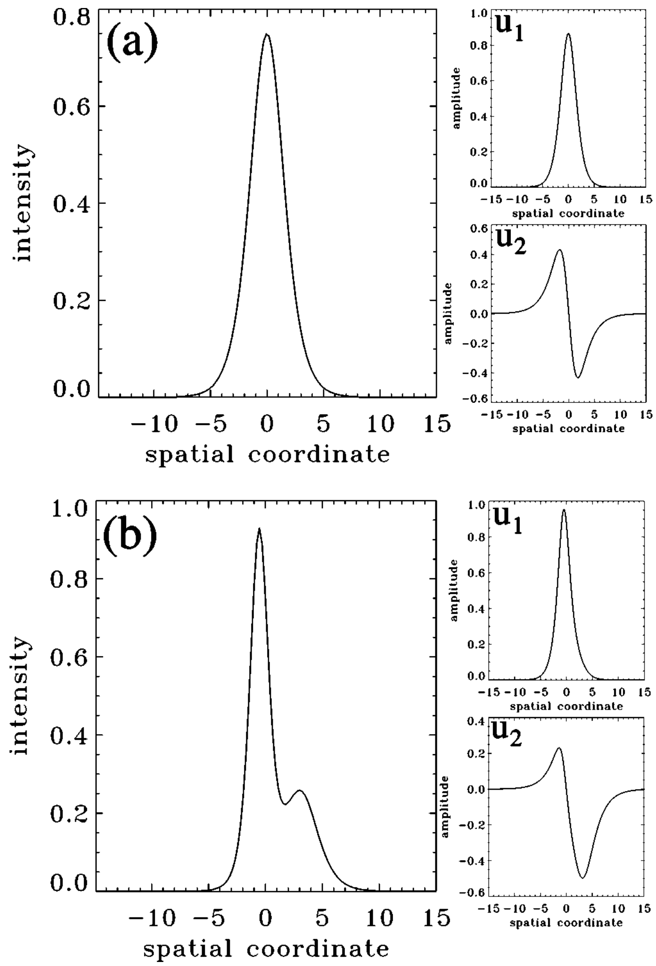

FIG. 1. Transverse profiles and linear modes of the PCS for $N=2$. Calculations use $k_1=1.0$, $k_2=0.5$. For symmetric solution (a), $\Delta x_{12}=0$, and for asymmetric solution (b), $\Delta x_{12}=2.0$.

V. GENERAL SOLUTION FOR $N=3$

If $N=3$, the coefficients a_j are

$$a_1=c_{12}c_{13}, \quad a_2=c_{12}c_{23}, \quad a_3=c_{13}c_{23}. \tag{16}$$

Note that $a_1=a_2-a_3+1$.

The explicit solution set for the arbitrary eigenvalue ($k_1>k_2>k_3$) case has the form

$$u_1(x)=\frac{2k_1\sqrt{a_1}}{D_3}[\cosh(k_2\bar{x}_2+k_3\bar{x}_3)+c_{23}\cosh(k_2\bar{x}_2-k_3\bar{x}_3)],$$

$$u_2(x)=\frac{2k_2\sqrt{a_2}}{D_3}[\sinh(k_1\bar{x}_1+k_3\bar{x}_3)+c_{13}\sinh(k_1\bar{x}_1-k_3\bar{x}_3)], \tag{17}$$

$$u_3(x)=\frac{2k_3\sqrt{a_3}}{D_3}[\cosh(k_1\bar{x}_1+k_2\bar{x}_2)-c_{12}\cosh(k_1\bar{x}_1-k_2\bar{x}_2)],$$

where

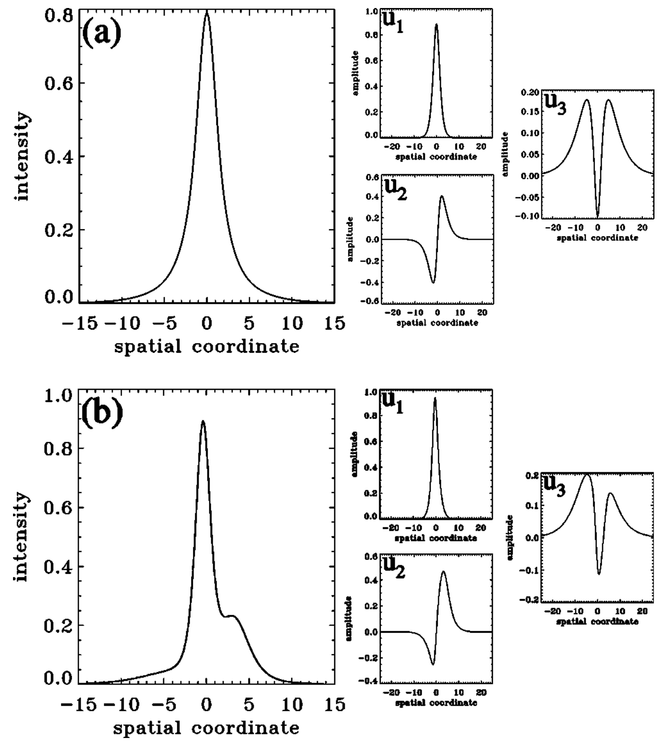

FIG. 2. Transverse profiles and linear modes of the PCS for $N=3$. Calculations use $k_1=1.0$, $k_2=0.5$, $k_3=0.2$. $\Delta x_{12}=\Delta x_{13}=0$ for symmetric solution (a) and $\Delta x_{12}=1.5$, $\Delta x_{13}=-1.0$ for asymmetric solution (b).

$$\begin{aligned}D_3(x)=&\cosh(k_1\bar{x}_1+k_2\bar{x}_2+k_3\bar{x}_3)\\&+a_1\cosh(k_1\bar{x}_1-k_2\bar{x}_2-k_3\bar{x}_3)\\&+a_2\cosh(k_1\bar{x}_1-k_2\bar{x}_2+k_3\bar{x}_3)\\&+a_3\cosh(k_1\bar{x}_1+k_2\bar{x}_2-k_3\bar{x}_3).\end{aligned} \tag{18}$$

This solution describes both symmetric and asymmetric functions. Note that $u_1(x)$ is a positive definite (nodeless) function, and thus is the ‘‘fundamental’’ mode. Examples giving two sets of parameters which lead to symmetric and asymmetric solutions are given in Figs. 2(a) and 2(b), respectively. To make the solutions symmetric, we have to set all $x_i=0$. Even in this case, the solution is still quite general and the intensity profile may have a complicated shape including double and triple peak structures. There is a special subclass of single peak symmetric solutions having $k_1=3k_3, k_2=2k_3$, $k_3=$arbitrary. In this particular case we find that D_3 reduces to $D_3=32\cosh^6(k_3x)$ and that the sum of the intensities is $\Sigma_{n=1}^{3}u_n^2(x)=6k_3^2\,\mathrm{sech}^2(k_3x)$. The components then agree with those which will be found in Sec. VII.

VI. GENERAL SOLUTION FOR $N=4$

Our calculations allow us to present the explicit symmetric solution set for the arbitrary eigenvalue ($k_1>k_2>k_3>k_4$) case:

$$u_1(x)=\frac{2k_1\sqrt{a_1}}{D_4}[\cosh(k_2\overline{x_2}+k_3\overline{x_3}+k_4\overline{x_4})$$
$$+c_{24}c_{34}\cosh(k_2\overline{x_2}+k_3\overline{x_3}-k_4\overline{x_4})$$
$$+c_{23}c_{34}\cosh(k_2\overline{x_2}+k_4\overline{x_4}-k_3\overline{x_3})$$
$$+c_{23}c_{24}\cosh(k_3\overline{x_3}+k_4\overline{x_4}-k_2\overline{x_2})],$$
$$u_2(x)=\frac{2k_2\sqrt{a_2}}{D_4}[\sinh(k_1\overline{x_1}+k_3\overline{x_3}+k_4\overline{x_4})$$
$$+c_{14}c_{34}\sinh(k_1\overline{x_1}+k_3\overline{x_3}-k_4\overline{x_4})$$
$$+c_{13}c_{34}\sinh(k_1\overline{x_1}+k_4\overline{x_4}-k_3\overline{x_3})$$
$$-c_{13}c_{14}\sinh(k_3\overline{x_3}+k_4\overline{x_4}-k_1\overline{x_1})], \tag{19}$$
$$u_3(x)=\frac{2k_3\sqrt{a_3}}{D_4}[\cosh(k_1\overline{x_1}+k_2\overline{x_2}+k_4\overline{x_4})$$
$$+c_{14}c_{24}\cosh(k_1\overline{x_1}+k_2\overline{x_2}-k_4\overline{x_4})$$
$$-c_{12}c_{24}\cosh(k_1\overline{x_1}+k_4\overline{x_4}-k_2\overline{x_2})$$
$$-c_{12}c_{14}\cosh(k_2\overline{x_2}+k_4\overline{x_4}-k_1\overline{x_1})],$$

and

$$u_4(x)=\frac{2k_4\sqrt{a_4}}{D_4}[\sinh(k_1\overline{x_1}+k_2\overline{x_2}+k_3\overline{x_3})$$
$$-c_{13}c_{23}\sinh(k_1\overline{x_1}+k_2\overline{x_2}-k_3\overline{x_3})$$
$$-c_{12}c_{23}\sinh(k_1\overline{x_1}+k_3\overline{x_3}-k_2\overline{x_2})$$
$$-c_{12}c_{13}\sinh(k_2\overline{x_2}+k_3\overline{x_3}-k_1\overline{x_1})],$$

where

$$D_4(x)=\cosh(k_1\overline{x_1}+k_2\overline{x_2}+k_3\overline{x_3}+k_4\overline{x_4})$$
$$+a_1\cosh(k_2\overline{x_2}+k_3\overline{x_3}+k_4\overline{x_4}-k_1\overline{x_1})$$
$$+a_2\cosh(k_3\overline{x_3}+k_4\overline{x_4}+k_1\overline{x_1}-k_2\overline{x_2})$$
$$+a_3\cosh(k_4\overline{x_4}+k_1\overline{x_1}+k_2\overline{x_2}-k_3\overline{x_3})$$
$$+a_4\cosh(k_1\overline{x_1}+k_2\overline{x_2}+k_3\overline{x_3}-k_4\overline{x_4})$$
$$+b_1\cosh(k_1\overline{x_1}+k_3\overline{x_3}-k_2\overline{x_2}-k_4\overline{x_4})$$
$$+b_2\cosh(k_1\overline{x_1}+k_2\overline{x_2}-k_3\overline{x_3}-k_4\overline{x_4})$$
$$+b_3\cosh(k_1\overline{x_1}+k_4\overline{x_4}-k_2\overline{x_2}-k_3\overline{x_3}). \tag{20}$$

Here we have used the convenient definitions [see Eq. (10)]

$$a_1=c_{12}c_{13}c_{14}, \quad a_2=c_{12}c_{23}c_{24},$$

and

$$a_3=c_{13}c_{23}c_{34}, \quad a_4=c_{14}c_{24}c_{34}, \tag{21}$$

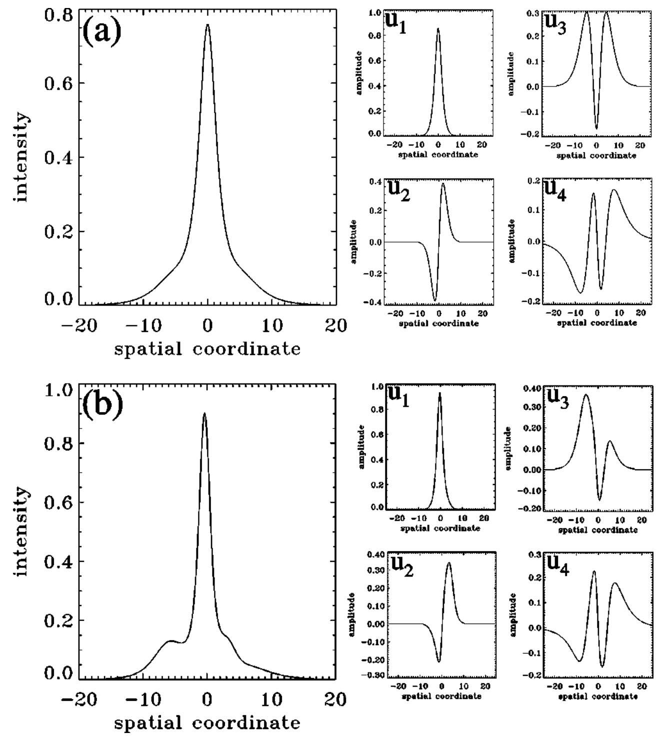

FIG. 3. Transverse profiles and linear modes of the PCS for $N=4$. Calculations use $k_1=1.0$, $k_2=0.6$, $k_3=0.4$, $k_4=0.2$. $\Delta x_{12}=\Delta x_{13}=\Delta x_{14}=0$ for symmetric solution (a) and $\Delta x_{12}=1.5$, $\Delta x_{13}=-1.5$, $\Delta x_{14}=0.5$ for asymmetric solution (b).

while

$$b_1=c_{12}c_{14}c_{23}c_{34}, \quad b_2=c_{13}c_{14}c_{23}c_{24},$$
$$b_3=c_{12}c_{13}c_{24}c_{34}. \tag{22}$$

We note that $a_1+a_3=a_2+a_4$ and $b_2+b_3=b_1+1$. We have used

$$c_{ij}=\frac{k_i+k_j}{k_i-k_j}, \tag{23}$$

where $i<j$ due to the ordering of the eigenvalues. Again, $u_1(x)$ has no zeros.

This solution also describes both symmetric and asymmetric PCSs. Figure 3 shows intensity profiles as well as the mode structure of these solutions for two sets of parameters, one of which leads to symmetric [Fig. 3(a)] and the other one to asymmetric [Fig. 3(b)] solutions. To make the solutions symmetric, we have to set all $x_i=0$. The symmetric solution still admits arbitrary k_i and the actual shape of PCS can be quite complicated with up to four peaks. For the special subclass of single peak symmetric solutions with $k_1=4k_4, k_2=3k_4, k_3=2k_4$, k_4 arbitrary, we see that D_4 reduces to $D_4=512\cosh^{10}(k_4x)$ and that the sum of the intensities is $\Sigma_{n=1}^4 u_n^2(x)=10k_4^2\,\mathrm{sech}^2(k_4x)$.

Generalization of the solutions to any $N>4$ is lengthy but straightforward. In each of these three sections ($N=1,2,3,4$) the special subclass has determinant $D_N=\frac{1}{2}[2\cosh(k_Nx)]^{N(N+1)/2}$ and the sum of intensities is $\Sigma u_n^2=(N/2)(N+1)k_N^2\,\mathrm{sech}^2(k_Nx)$. In the next section, we will further investigate these special solutions.

VII. SYMMETRIC SOLUTIONS IN TERMS OF ASSOCIATED LEGENDRE FUNCTIONS

Suppose that the eigenvalues are equally spaced, i.e., $k_1 = Nk_N$, $k_2=(N-1)k_N$, up to $k_{N-1}=2k_N$ and k_N, where k_N is arbitrary. This is a special case of general solution found above. This solution set is based on the modes of the ''sech-squared'' waveguide [32]. The additional condition here is that each x_i is zero. The above choices obey the condition that the sum of the components equals a fixed multiple of the function $\mathrm{sech}^2(k_N x)$. Hence, each component must satisfy

$$u_n''(x)+k_N^2 N(N+1)\mathrm{sech}^2(k_N x)u_n(x) = k_N^2(N-n+1)^2 u_n(x), \quad n=1,2,\ldots,N. \tag{24}$$

Using $\xi=\tanh(k_N x)$ transforms each equation of this form into the differential equation for the associated Legendre functions, so the solutions can be written in terms of the associated Legendre functions [32]:

$$u_n(x)=\pm c_n P_N^{(N-n+1)}(\xi)=\pm c_n P_N^{(N-n+1)}[\tanh(k_N x)], \tag{25}$$

where the c_n are constants. These functions can be written explicitly using [33]

$$P_N^j(y)=\mathrm{sech}^j(y)\left(\frac{d^j}{d\xi^j}P_N(\xi)\right), \quad \xi=\tanh(y) \tag{26}$$

where P_N is the Legendre polynomial of order N. To satisfy the original equation set, we need

$$2\sum_{n=1}^{N} u_n^2(x)=k_N^2 N(N+1)\mathrm{sech}^2(k_N x), \tag{27}$$

i.e.,

$$2\sum_{n=1}^{N} c_n^2[P_N^{N-n+1}(\xi)]^2=k_N^2 N(N+1)(1-\xi^2). \tag{28}$$

Note that each intensity u_n^2 is a polynomial in ξ. Equating the coefficients of the polynomials then provides the c_n. Of course each component can be independently multiplied by ± 1 as the index only involves intensities. Thus for $N=2$ we obtain $c_1=\pm c_2=\pm k_2/\sqrt{3}$. The solution with no nodes always has $n=1$, so the ordering agrees with that of the previous sections. Then $u_2(x)$ has one node, $u_3(x)$ has two nodes, etc., and finally $u_N(x)$ has $N-1$ nodes.

For example, for $N=2$, we have

$$u_1(x)=\sqrt{3}k_2\,\mathrm{sech}^2(k_2 x), \tag{29}$$

$$u_2(x)=\sqrt{3}k_2\,\mathrm{sech}(k_2 x)\tanh(k_2 x). \tag{30}$$

For the $N=3$ case, we have $k_1=3k_3$, $k_2=2k_3$ while k_3 is arbitrary; the solution is

$$u_1(x)=\frac{3}{4}\sqrt{10}k_3\,\mathrm{sech}^3(k_3 x), \tag{31}$$

$$u_2(x)=\sqrt{15}k_3\,\mathrm{sech}^2(k_3 x)\tanh(k_3 x), \tag{32}$$

$$u_3(x)=\frac{3k_3}{2\sqrt{6}}\mathrm{sech}(k_3 x)[1-5\tanh^2(k_3 x)]. \tag{33}$$

In general, the lowest-order function, $u_1(x)$, is proportional to $\mathrm{sech}^N(k_N x)$ and is symmetric. $u_j(x)$ is symmetric if j is odd and is antisymmetric if j is even.

The condition of Eq. (27) specifies the coefficients of the associated Legendre functions. The solution set for $N=4$ is

$$u_1(x)=\frac{k_4}{2}\sqrt{35}\,\mathrm{sech}^4(k_4 x), \tag{34}$$

$$u_2(x)=\frac{3}{2}\sqrt{\frac{35}{2}}k_4\,\mathrm{sech}^3(k_4 x)\tanh(k_4 x), \tag{35}$$

$$u_3(x)=\frac{k_4}{2}\sqrt{5}\,\mathrm{sech}^2(k_4 x)[7\tanh^2(k_4 x)-1], \tag{36}$$

$$u_4(x)=\frac{5k_4}{2\sqrt{10}}\mathrm{sech}(k_4 x)\tanh(k_4 x)[7\tanh^2(k_4 x)-3]. \tag{37}$$

These solutions have been presented in Refs. [13,14] in relation to PCSs.

VIII. COLLISIONS

The most intriguing property of PCSs is their collision behavior. Having analytical solutions to this interesting problem allows us to describe the changes which multisoliton complexes undergo after collisions. It can be shown [27] that multisoliton solutions of Eq. (1) are simultaneously solutions of the linear set of algebraic functional equations:

$$\sum_{i=1}^{N}\frac{\exp(k_i^*\overline{x_i}-ik_i^{*2}z/2)\exp(k_j\overline{x_j}+ik_j^2 z/2)}{k_j+k_i^*}\frac{\psi_i(x,z)}{\sqrt{2\,\mathrm{Re}\,k_i}} + \frac{\psi_j(x,z)}{\sqrt{2\,\mathrm{Re}\,k_j}} = -\frac{\exp(k_j\overline{x_j}+ik_j^2 z/2)}{\sqrt{\alpha}}, \tag{38}$$

where the eigenvalues are $k_j=\lambda_j+iV_j$, with λ_j being the amplitude and $V_j=\tan\theta_j$ being the velocity of each soliton, and with each x_j being related to the initial location of the soliton center. When dealing with two colliding PCSs, we have to choose a different value of V_j for each PCS. Hence, we have only two values of V. This fact somewhat simplifies the calculations.

Equations (38) can be written in a matrix form:

$$D_{j,m}\frac{\psi_m(x,z)}{\sqrt{2\,\mathrm{Re}\,k_m}}=-\frac{e_j}{\sqrt{\alpha}}, \tag{39}$$

where the Hermitian matrix D,

$$D_{j,m}=\delta_{j,m}+\frac{e_j e_m^*}{k_j+k_m^*}, \tag{40}$$

has a real determinant and

$$e_j = \exp(k_j \bar{x}_j + i k_j^2 z/2). \tag{41}$$

The multisoliton solutions of the generalized Manakov equations can be written in the form

$$\psi_i = -\frac{D_{ij}^{-1}\sqrt{2\,\mathrm{Re}\,k_j}\,e_j}{\sqrt{\alpha}}. \tag{42}$$

The solution describes both the form of partially coherent solitons of any order and their collisions. The best way to use the solution (42) is numerical inversion of the matrix because the analytical expressions are not as simple as those for PCSs. It is easy to show that for $N=1$, the solution is a soliton of a single nonlinear Schrödinger equation. Otherwise, the solution is a superposition of single soliton components.

Suppose we have two PCSs with components N_1 and N_2, respectively, such that $N_1+N_2=N$. Let us consider collisions between them. The interpretation of a PCS as a multisoliton complex suggests that collisions must reshape them. In fact, the N eigenvalues k_i must be conserved during the collision, but the $N-1$ relative separations, Δx_{ij}, must change. As a result, the shapes of PCSs do not have to be preserved. Equation (42) allows us to calculate the asymptotic values for fundamental solitons after the collision and, as a result, the change in their relative separations. This change can also be calculated using the Manakov result [34] for pairwise collisions. All lateral shifts are additive quantities. Adding up the shifts of the individual collisions gives the total collision-induced shift for the ith soliton in the first PCS:

$$\delta x_i = \frac{1}{\lambda_i}\sum_{k=N_1+1}^{N} \ln\sqrt{\frac{(\tan\theta_1 - \tan\theta_2)^2 + (\lambda_i+\lambda_k)^2}{(\tan\theta_1 - \tan\theta_2)^2 + (\lambda_i-\lambda_k)^2}},$$

$$i = 1,2,3,\ldots,N_1 \tag{43}$$

where θ_1 and θ_2 are the angles of incidence of each of the two PCSs. A similar expression can be written for soliton shifts in the second PCS. Clearly, these shifts are different for each soliton component in a given PCS. The net result is PCS reshaping. We should also mention that, because of the integrability of the model, collisions are elastic and radiation waves are not created. The output consists only of the reshaped PCS, but it contains no radiation.

Examples of collisions are presented in Figs. 4–6. The plots have been obtained using Eq. (42). A collision of two PCSs, each consisting of three fundamental solitons, is shown in Fig. 4. The main feature of the collisions is that the PCS remains and propagates as a stationary solution after the collision, but its shape changes. As we explained above, the reason for this is that the N partial amplitudes in the PCS do not change during interactions, but the relative separations, $x_i - x_j$, of the constituent solitons do change. The net result is a restructuring of the PCS after a collision.

A spectacular example of this rearrangement is shown in Fig. 5. In this case, each PCS is initially slightly asymmetric and consists of six linear modes. Because of the multiple interactions, the shifts [Eq. (43)] are large, and as a result each output beam is almost completely separated into its six fundamental solitons. We can think of this process as the transformation and a separation of a partially coherent beam into its coherent components. This idea could be used in some applications where an initially incoherent wave packet can be transformed into a number of coherent beams. Interestingly enough, the larger the number of components, the larger is their separation after the collision.

One more example of a soliton collision is shown in Fig. 6. This is a very special case of a collision between a fundamental soliton and a PCS consisting of two modes. Evidently, in this case, the shape of the PCS soliton *does not change* after the collision. This may seem to be rather surprising in the light of our earlier discussion concerning the

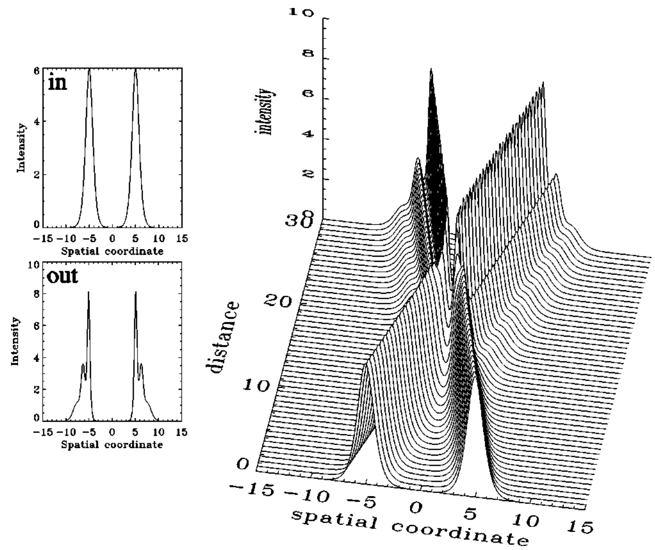

FIG. 4. Collision of two symmetric PCSs, each consisting of three modes. The inset shows the input and output profiles of the PCSs. Parameters chosen in this simulation are $\lambda_1=3.0$, $\lambda_2=2.0$, $\lambda_3=1.0$, $\Delta x_{12}=\Delta x_{13}=0$ and the angle of collision is such that $\tan\theta=0.3$.

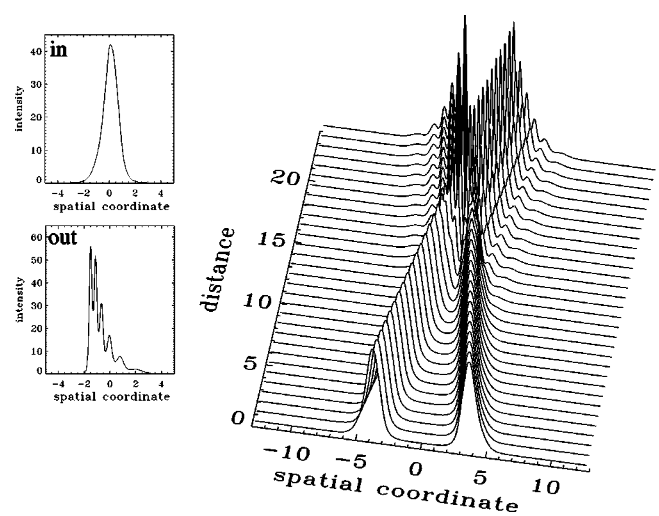

FIG. 5. Collision of two slightly asymmetric PCSs, each consisting of six linear modes. The inset shows the input and output profiles of the PCS. Parameters chosen in this simulation are $\lambda_1=6.0$, $\lambda_2=5.0$, $\lambda_3=4.0$, $\lambda_4=3.0$, $\lambda_5=2.0$, $\lambda_6=1.0$, $\Delta x_{12}=0$, $\Delta x_{13}=-0.2$, $\Delta x_{14}=-0.1$, $\Delta x_{15}=-0.3$, $\Delta x_{16}=-0.1$ and the angle of collision is chosen such that $\tan\theta=0.3$.

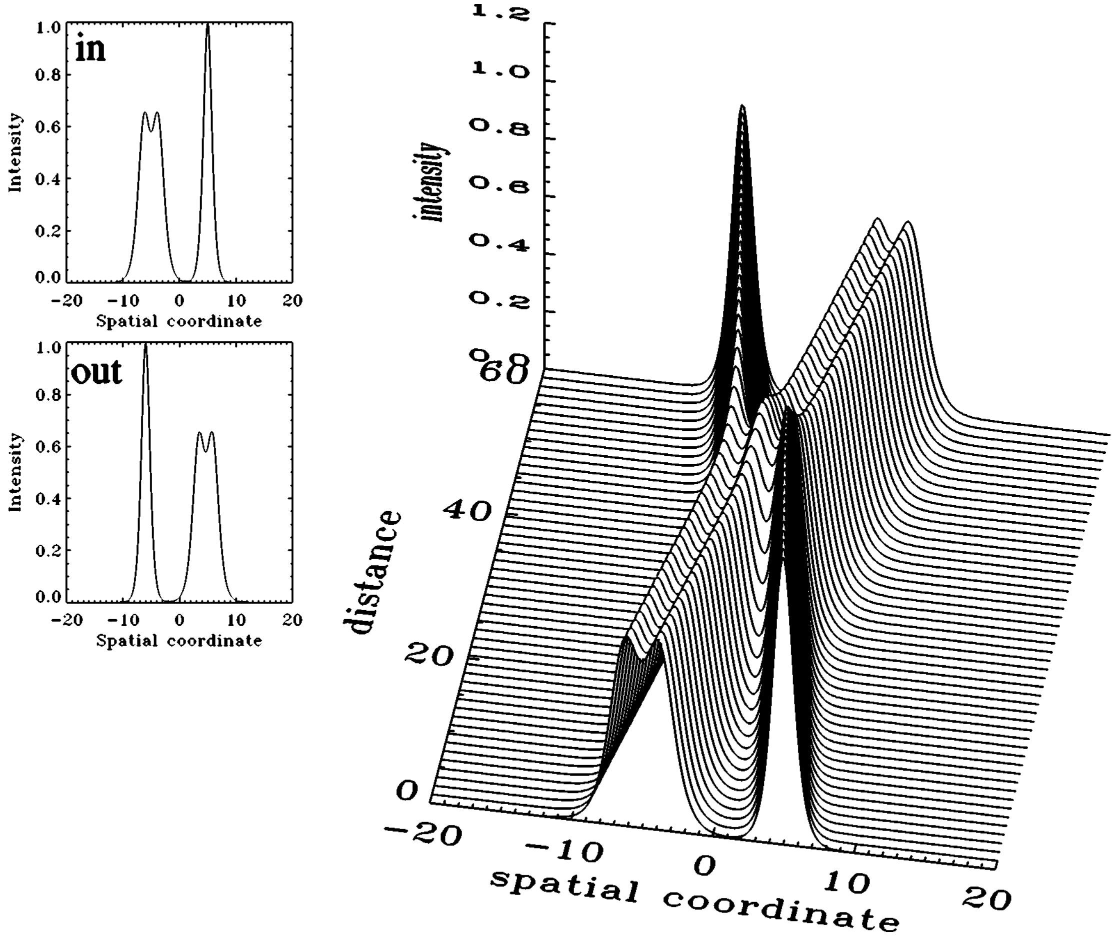

FIG. 6. Collision of PCS formed by two linear modes with a single soliton. The inset shows the initial and output intensity profiles of both solitons. Parameters chosen in this simulation are $\lambda_1=1.0$, $\lambda_2=0.65$, $\Delta x_{12}=0$ (for the PCS), $\lambda_3=1.0$ (for the single soliton). The angle of collision is such that $\tan\theta=0.1253$. Note that the profile of the PCS does not change after the collision. This happens only at this specific angle of collision when the two constituent solitons which form the PCS experience exactly the same lateral shift during collision so that the final separation again is $\Delta x_{12}=0$.

collisions of partially coherent solitons. However, the explanation of this result is very simple. The partially coherent soliton does not change its shape when all of its components experience exactly the same lateral shift during the collision. This is possible because the lateral shift of each soliton is a nonlinear function of its amplitude and velocity. It can be shown that, for a situation like that in Fig. 6, there exists a collision angle for which $\delta x_1=\delta x_2$, and, consequently, the profile of the soliton remains unchanged. It should be stressed, though, that this is only possible for very specific parameters of the collision. In the general case, the shape of a PCS changes dramatically.

IX. CONCLUSIONS

We have found exact solutions of the generalized Manakov equations which describe partially coherent solitons and their collisions. The exact solutions allowed us to find stationary profiles of the spatial beams and predict the result of their collisions. We have found, analytically, the number of parameters that control the soliton shape. We have found profiles which are asymmetric in general, but which become symmetric for certain values of the parameters. We have also found that collisions allow the profiles to remain stationary but change their shapes substantially.

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